Variational methods, linearization tools, and symmetrization for spectral problems with the p-Laplacian
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ABSTRACT. We look for weak solutions $u \in W_0^{1,p}(\Omega)$ of the degenerate quasilinear Dirichlet boundary value problem

$$\text{(P)} \qquad -\Delta_p u = \lambda |u|^{p-2} u + f(x, u(x)) \ \text{ in } \Omega\,; \qquad u = 0 \ \text{ on } \partial\Omega\,.$$

It is assumed that $1 < p < \infty$, $p \neq 2$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian, Ω is a bounded domain in $\mathbb{R}^N$, $f(\,\cdot\,, u) \in L^\infty(\Omega)$ is a given function of $u \in \mathbb{R}$, and λ stands for the (real) spectral parameter. If $f(x, u) \equiv f(x)$ is independent from $u \in \mathbb{R}$, problem (P) is closely connected with the Fredholm alternative for the $(p-1)$-homogeneous quasilinear operator $-\Delta_p$ on $W_0^{1,p}(\Omega)$. Such weak solutions are precisely the critical points of the corresponding energy functional on $W_0^{1,p}(\Omega)$,

$$\text{(J)} \qquad \mathcal{J}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p} \int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f(x)\, u \,\mathrm{d}x\,, \quad u \in W_0^{1,p}(\Omega)\,.$$

I.e., problem (P) is equivalent with $\mathcal{J}'_\lambda(u) = 0$ in $W^{-1,p'}(\Omega)$. Here, $\mathcal{J}'_\lambda(u)$ stands for the (first) Fréchet derivative of the functional $\mathcal{J}_\lambda$ on $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ denotes the (strong) dual space of the Sobolev space $W_0^{1,p}(\Omega)$, $p' = p/(p-1)$.

We will describe a global minimization method for this functional provided $\lambda < \lambda_1$, together with the (strict) convexity of the functional for $\lambda \leq 0$ and possible "nonconvexity" if $0 < \lambda < \lambda_1$. As usual, λ_1 denotes the first (smallest) eigenvalue λ_1 of the positive p-Laplacian $-\Delta_p$. Strict convexity will force the uniqueness of a critical point (which is then the global minimizer for $\mathcal{J}_\lambda$), whereas "nonconvexity" will be shown by constructing a saddle point which is different from any local or global minimizer.

Finally, we will discuss the *existence* and *multiplicity* of a solution for problem (P) when $f(x, u)$ is decreasing in u. We will describe local and global minimization methods for the corresponding functional.

Keywords: nonlinear eigenvalue problem; Fredholm alternative; degenerate or singular quasilinear Dirichlet problem; p-Laplacian; global minimizer; minimax principle

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1 Lecture 1: The Riesz representation theorem in $L^p(\Omega)$

2 Lecture 2: The energy functional – convex / concave, convexity and uniqueness

3 Lecture 3: The (first and second) eigenvalues of $-\Delta_p$

4 Lecture 4: Nonconvex energy functional – minimization with constraint

5 Lecture 5: Existence by a topological degree